Eigenfunctions of hermitian Operators

Note Title

I. Discrete Spectra

1. Their eigenvalues are real

$$|\hat{Q}\psi\rangle = q|\psi\rangle$$

If $\hat{Q}$ is hermitian

$$\langle\psi|\hat{Q}\psi\rangle = \langle\hat{Q}\psi|\psi\rangle$$

Left side : $\langle\psi|\hat{Q}\psi\rangle = q\langle\psi|\psi\rangle$

Right side : $\langle\hat{Q}\psi| = (|Q\psi\rangle)^\dagger = (q|\psi\rangle)^\dagger$

$$= q^*\langle\psi|$$

$$\langle\hat{Q}\psi|\psi\rangle = q^*\langle\psi|\psi\rangle$$

Thus $q = q^*$ $\Rightarrow$ q must be real

2. Eigenfunctions belonging to different eigenvalues are orthogonal

$$\hat{Q}|f\rangle = q|f\rangle \Rightarrow \langle Qf| = q\langle f|$$

$$\hat{Q}|g\rangle = q'|g\rangle$$

$$\rightarrow \langle f|\hat{Q}g\rangle = q'\langle f|g\rangle$$

$$\rightarrow \langle Qf|g\rangle = q\langle f|g\rangle$$

Since the left sides are equal to each other, $q'\langle f|g\rangle = q\langle f|g\rangle$

$$\Rightarrow (q'-q)\langle f|g\rangle = 0$$

$$\Rightarrow \text{If } q' \neq q, \quad \langle f|g\rangle = 0$$

i.e. f and g are orthogonal.

* If two or more linearly independent eigenfunctions share the same eigenvalue, the spectrum is said to be degenerate.

* Even if two or more eigenfunctions share the same eigenvalue, it is always possible to construct orthogonal functions out of them.

II. Continuous spectra

In this case, the eigenfunctions are not normalizable.

But they still satisfy the so-called "Dirac orthonormality" such that

$$\langle f_{p'} | f_p \rangle = \delta(p - p')$$

[compare to the discrete case

$$\langle f_m | f_n \rangle = \delta_{mn}$$]

this $\hat{p}$ is an operator

this p is a number

$\boxed{\text{Ex 1}}$ Eigenftns and eigenvalues for the momentum operator $\hat{p}$

$$\hat{p} f_p(x) = p f_p(x) \Rightarrow \frac{\hbar}{i}\frac{d}{dx} f_p(x) = p f_p(x)$$

$$\Rightarrow \frac{d f_p(x)}{dx} = i\frac{p}{\hbar} f_p(x) \rightarrow f_p(x) = A e^{i\frac{p}{\hbar}x}$$

$\langle f_{p'} | f_p \rangle$

$$= \int_{-\infty}^{\infty} f_{p'}^*(x)\, f_p(x)\, dx$$

$$= |A|^2 \int_{-\infty}^{\infty} e^{-i\frac{p'}{\hbar}x}\, e^{i\frac{p}{\hbar}x}\, dx$$

$$= |A|^2 \int_{-\infty}^{\infty} e^{i\frac{p-p'}{\hbar}x}\, dx$$

[From G. Prob. 2.26 $\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\, dk$]

$$= |A|^2 \int_{-\infty}^{\infty} e^{i(p-p')\frac{x}{\hbar}}\, d\left(\frac{x}{\hbar}\right) \cdot \hbar$$

$$= 2\pi\hbar |A|^2\, \delta(p-p')$$

Thus if we take $A = \frac{1}{\sqrt{2\pi\hbar}}$

$$\langle f_{p'} | f_p \rangle = \delta(p-p')$$

with the eigenfunction $f_p(x) = \dfrac{e^{i\frac{p}{\hbar}x}}{\sqrt{2\pi\hbar}}$

* Any functions in the Hilbert space can be expressed as a linear combination of the eigenftns of an observable operator. We say that these eigenftns are "complete".

* Although the eigenfunctions of continuous spectrum are not in the Hilbert space, because they are not exactly normalizable (i.e. $\langle f_p | f_p \rangle = \delta(0) = \infty$), they are still complete.

In other words, any arbitrary function in the Hilbert space, $f(x)$, can be expressed as a linear combination of these eigenfunctions such that

$$f(x) = \int_{-\infty}^{\infty} c(p) f_p(x) \, dp$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} c(p) \, e^{ipx/\hbar} \, dp$$

And

$$\underline{\langle f_{p'} | f \rangle} = \int_{-\infty}^{\infty} c(p) \langle f_{p'} | f_p \rangle \, dp$$

$$= \int_{-\infty}^{\infty} c(p) \delta(p - p') \, dp$$

$$= \underline{c(p')}$$

That is, $c(p) = \langle f_p | f \rangle = \int_{-\infty}^{\infty} f_p^*(x) f(x) \, dx$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-i\frac{px}{\hbar}} f(x) \, dx$$

Recall for the discrete case,

$$f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$$

$$c_n = \langle f_n | f \rangle = \int_{-\infty}^{\infty} f_n^*(x) f(x) dx$$

$\boxed{\text{Ex. 2}}$ Eigenftns and eigenvalues of the position operator.

As we did with the $\hat{p}$ operator

: $\hat{p} f_p(x) = p f_p(x)$ ----

$$\hat{x} g_y(x) = y g_y(x)$$

a number

an operator, which is just x, a continuous variable in the position space

$\Rightarrow$ $(x-y) g_y(x) = 0$ $\Rightarrow \begin{cases} g_y(x) = 0 & \text{for } x \neq y \\ g_y(x) \neq 0 & \text{for } x = y \end{cases}$

$\therefore$ $g_y(x) = A\,\delta(x-y)$

Now Dirac orthonormality requires

$$\delta(y-y') = \langle g_{y'} | g_y \rangle = \int_{-\infty}^{\infty} g_{y'}^*(x) g_y(x) dx$$

$$= |A|^2 \int_{-\infty}^{\infty} \delta(x-y') \delta(x-y) dx$$

$$= |A|^2 \delta(y-y') \Rightarrow A = 1$$

$\therefore \quad g_y(x) = \delta(x-y)$

They are also complete, since any arbitrary ftn in the Hilbert space,

$$f(x) = \int_{-\infty}^{\infty} c(y)\, g_y(x)\, dy$$

$$\Rightarrow \quad c(y) = \langle g_y | f \rangle = \int g_y^*(x) f(x)\, dx$$
$$= \int \delta(x-y) f(x)\, dx = f(y)$$

In other words, $f(x) = \int_{-\infty}^{\infty} f(y)\, \delta(x-y)\, dy$

, a well-known trivial result.

Generalized Statiscal Interpretation

* If we measure an observable $Q(x, p)$ on a particle in the state $\Psi(x,t)$, the probability of measuring a particular discrete eigen value q_n with the corresponding eigenft $f_n(x)$ is $\quad |c_n = \langle f_n | \Psi \rangle|^2$

For a continuous spectrum, the probability of measuring an eigenvalue between q and $q+dq$ with the eigenfunction $f_q(x)$ is

$$|c(q) = \langle f_q | \Psi \rangle|^2 dq$$

* The eigenftns are complete, Thus

$$\Psi(x,t) = \sum_n c_n(t) f_n(x)$$

$$\Rightarrow c_n(t) = \langle f_n | \Psi \rangle$$

$$= \int f_n^*(x) \Psi(x,t) dx$$

For an observable Q, for which $Qf_n = q_n f_n$,

$$\langle Q \rangle = \sum_n q_n |c_n|^2.$$

* For the continuous spectrum of q

$$\Psi(x,t) = \int c(q,t) f_q(x) dq$$

$$\Rightarrow c(q,t) = \langle f_q | \Psi \rangle$$

$$= \int f_q^*(x) \Psi(x,t) dx$$

For the special case of the momentum operator, (i.e. $q \Rightarrow p$),

we express $c(q,t)$ by $\Phi(p,t)$ and call it the "momentum space wave function", whereas the usual wave function $\Psi(x,t)$ is called the "position space" wave function.

In other words

$$\Phi(p,t) = \langle f_p | \Psi \rangle$$

$$= \int f_p^*(x)\, \Psi(x,t)\, dx$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-i\frac{p}{\hbar}x}\, \Psi(x,t)\, dx$$

$$\Psi(x,t) = \int \Phi(p,t)\, f_p(x)\, dp$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \Phi(p,t)\, e^{i\frac{p}{\hbar}x}\, dp$$

Note that this is a generalized expression of the free particle result of chap. 2

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{i(kx - \frac{E_k}{\hbar}t)}\, dk$$

The probability of measuring the momentum between p_1 and p_2 is given by

$$\int_{p_1}^{p_2} |\Phi(p,t)|^2 dp$$

, just as the probability of finding the position between x_1 and x_2 is

$$\int_{x_1}^{x_2} |\Psi(x,t)|^2 dx$$

* Also expectation value of any operator $Q(x,p)$ can be evaluated both in position space and momentum space, such that

$$\langle Q(x,p) \rangle = \begin{cases} \int \Psi^* Q(x, \frac{\hbar}{i}\frac{\partial}{\partial x})\, \Psi\, dx \\ \int \Phi^* Q(-\frac{\hbar}{i}\frac{\partial}{\partial p}, p)\, \Phi\, dp \end{cases}$$